# NECKLACE LIE ALGEBRAS AND NONCOMMUTATIVE SYMPLECTIC GEOMETRY

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Abstract. Recently, V. Ginzburg proved that Calogero phase space is a coadjoint orbit for some infinite dimensional Lie algebra coming from noncommutative symplectic geometry, [8]. In this note we generalize his argument to specific quotient varieties of representations of (deformed) preprojective algebras. This result was also obtained independently by V. Ginzburg [9]. Using results of W. Crawley-Boevey and M. Holland [6], [4] and [5] we give a combinatorial description of all the relevant couples $(\alpha, \lambda)$ which are coadjoint orbits. Finally we explain the coadjoint settings as those for which there is a Cayley-smooth algebra associated to them.

## 1. Introduction.

Over the last couple of years some surprising new results were obtained about the space $Weyl$ of isomorphism classes of right ideals in the first Weyl algebra $A_1(\mathbb{C})$ linking its structure to the adelic Grassmannian $Gr^{ad}$ and certain moduli spaces.

Let $\lambda \in \mathbb{C}$, a subset $V \subset \mathbb{C}[x]$ is said to be $\lambda$-*primary* if there is some power $r \in \mathbb{N}_+$ such that

$$(x-\lambda)^r \mathbb{C}[x] \subset V \subset \mathbb{C}[x]$$

A subset $V \subset \mathbb{C}[x]$ is said to be *primary decomposable* if it is the finite intersection

$$V = V_{\lambda_1} \cap \ldots \cap V_{\lambda_r}$$

with $\lambda_i \neq \lambda_j$ if $i \neq j$ and $V_{\lambda_i}$ is a $\lambda_i$-primary subset. Let $k_{\lambda_i}$ be the codimension of $V_{\lambda_i}$ in $\mathbb{C}[x]$ and consider the polynomial

$$p_V(x) = \prod_{i=1}^{r} (x-\lambda_i)^{k_{\lambda_i}}$$

Finally, take $W = p_V(x)^{-1}V$, then $W$ is a vectorsubspace of the rational function-field $\mathbb{C}(x)$ in one variable.

**Definition 1.1.** The *adelic Grassmannian* $Gr^{ad}$ is the set of subspaces $W \subset \mathbb{C}(x)$ that arise in this way.

We can decompose $Gr^{ad}$ in affine cells as follows. For a fixed $\lambda \in \mathbb{C}$ we define

$$Gr_\lambda = \{W \in Gr^{ad} \ \mid \ \exists k, l \in \mathbb{N} \ : \ (x-\lambda)^k \mathbb{C}[x] \subset W \subset (x-\lambda)^{-l} \mathbb{C}[x]\}$$

Then, we can write every element $w \in W$ as a *Laurent series*

$$w = \alpha_s (x-\lambda)^s + \text{ higher terms}$$

Consider the increasing set of integers $S = \{s_0 < s_1 < \ldots\}$ consisting of all *degrees* $s$ of elements $w \in W$. Now, define natural numbers

$$v_i = i - s_i \qquad \text{then} \qquad v_0 \geq v_1 \geq \ldots \geq v_z = 0 = v_{z+1} = \ldots$$

That is, to $W \in Gr_\lambda$ we can associate a *partition*

$$p(W) = (v_0, v_1, \ldots, v_{z-1})$$

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Conversely, if $p$ is a partition of some $n$, then the set of all $W \in Gr_\lambda$ with associated partition $p_W = p$ form an affine space $\mathbb{A}^n$ of dimension $n$. Hence, $Gr_\lambda$ has a cellular structure indexed by the set of all partitions.

As $Gr^{ad} = \prod'_{\lambda \in \mathbb{C}} Gr_\lambda$ because for any $W \in Gr^{ad}$ there are uniquely determined $W(\lambda_i) \in Gr_{\lambda_i}$ such that $W = W(\lambda_1) \cap \ldots \cap W(\lambda_r)$, there is a natural number $n$ associated to $W$ where $n = |p_i|$ where $p_i = p(W(\lambda_i))$ is the partition determined by $W(\lambda_i)$. Again, all $W \in Gr^{ad}$ with corresponding $(\lambda_1, p_1; \ldots; \lambda_r, p_r)$ for an affine cell $\mathbb{A}^n$ of dimension $n$. In his way, the adelic Grassmannian $Gr^{ad}$ becomes an infinite cellular space with the cells indexed by $r$-tuples of complex numbers and partitions for all $r \geq 0$.

A surprising connection between $Gr^{ad}$ and the Calogero system was discovered by G. Wilson in [20].

**Theorem 1.2.** *Let $Gr^{ad}(n)$ be the collection of all cells of dimension $n$ in $Gr^{ad}$, then there is a set-theoretic bijection*

$$Gr^{ad}(n) \longleftrightarrow Calo_n$$

*between $Gr^{ad}(n)$ and the phase space of $n$ Calogero particles, that is, the orbit space of couples of $n \times n$ matrices $(X, Y)$ such that the rank of $[X, Y] - \mathbb{1}_n$ is one under simultaneous conjugation.*

The connection between right ideals of $A_1(\mathbb{C})$ and $gr^{ad}$ is contained (in disguise) in the paper of R. Cannings and M. Holland [3]. $A_1(\mathbb{C})$ acts as differential operators on $\mathbb{C}[x]$ and for every right ideal $I$ of $A_1(\mathbb{C})$ they show that $I.\mathbb{C}[x]$ is primary decomposable. Conversely, if $V \subset \mathbb{C}[x]$ is primary decomposable, they associate the right ideal

$$I_V = \{\theta \in A_1(\mathbb{C}) \mid \theta.\mathbb{C}[x] \subset V\}$$

of $A_1(\mathbb{C})$ to it. Moreover, isomorphism classes of right ideals correspond to studying primary decomposable subspaces under multiplication with polynomials. Hence,

$$Gr^{ad} \simeq Weyl$$

The group $Aut\ A_1(\mathbb{C})$ of $\mathbb{C}$-algebra automorphisms of $A_1(\mathbb{C})$ acts on the set of right ideals of $A_1(\mathbb{C})$ and respects the notion of isomorphism whence acts on $Weyl$. The group $Aut\ A_1(\mathbb{C})$ is generated by automorphisms $\sigma_i^f$ defined by

$$\begin{cases} \sigma_1^f(x) &= x + f(y) \\ \sigma_1^f(y) &= y \end{cases} \quad \text{with } f \in \mathbb{C}[y], \quad \begin{cases} \sigma_2^f(x) &= x \\ \sigma_2^f(y) &= y + f(x) \end{cases} \quad \text{with } f \in \mathbb{C}[x]$$

For a natural number $n \geq 1$ we define the *$n$-th canonical right ideal* of $A_1(\mathbb{C})$ to be

$$\mathfrak{p}_n = x^{n+1} A_1(\mathbb{C}) + (xy + n) A_1(\mathbb{C}).$$

One can show that $\mathfrak{p}_n \not\simeq \mathfrak{p}_m$ whenever $n \neq m$ so the isomorphism classes $[\mathfrak{p}_n]$ are distinct points in $Weyl$ for all $n$. We define

$$Weyl_n = Aut\ A_1(\mathbb{C}).[\mathfrak{p}_n] = \{\ [\sigma(\mathfrak{p}_n)] \quad \forall \sigma \in Aut\ A_1(\mathbb{C})\}$$

the orbit in $Weyl$ of the point $[\mathfrak{p}_n]$ under the action of the automorphism group.

Yu. Berest and G. Wilson proved in [2] that the Cannings-Holland correspondence respects the automorphism orbit decomposition.

**Theorem 1.3.** *We have $Weyl = \bigsqcup_n Weyl_n$ and there are set-theoretic bijections*

$$Weyl_n \longleftrightarrow Gr^{ad}(n)$$

*whence also with $Calo_n$.*

An early indication that a decomposition into moduli spaces might be possible can be found in the paper [15] of the second author. Recently, a similar idea was pursued by A. Kapustin, A. Kutzetsov and D. Orlov [11].

If we trace the action of $Aut\ A_1(\mathbb{C})$ on $Weyl_n$ through all the identifications, we get a transitive action of $Aut\ A_1(\mathbb{C})$ on $Calo_n$. However, this action is non-differentiable hence highly non-algebraic. Berest and Wilson asked whether it is possible to identify $Calo_n$ with a coadjoint orbit in some infinite dimensional Lie algebra.

This conjecture was proved by V. Ginzburg [8] using noncommutative symplectic geometry as sketched by M. Kontsevich [14]. After reading his preprint it became clear to us that his method could be used almost verbatim for the quotient varieties of representations of deformed preprojective algebras. The two crucial steps are (1) invariants of quiver representations are generated by traces along oriented cycles, proved in [17] and (2) acyclicity of the relative (!) noncommutative deRham cohomology for path algebras of quivers.

In the first sections of this paper we carry out the second project in some detail. When applied to the double $\mathbb{Q}$ of a quiver, the noncommutative functions acquire the structure of an infinite dimensional Lie algebra $\mathbb{N}_Q$ which we call the necklace Lie algebra of a quiver. A major result asserts that this Lie algebra is a central extension of the Lie algebra of symplectic derivations, that is the Lie algebra corresponding to the vertex-preserving automorphisms of the path algebra $\mathbb{CQ}$ preserving the moment element $m = \sum_a [a, a^*]$.

Recall that the deformed preprojective algebra $\Pi_\lambda$ is defined to be the quotient of $\mathbb{CQ}$ by the twosided ideal generated by $m - \lambda$. Our generalization of Ginzburg's result on the coadjointness of Calogero space can then be stated as :

**Theorem 1.4.** *The variety $iss_\alpha\ \Pi_\lambda$ of isomorphism classes of semisimple $\alpha$-dimensional representations of the deformed preprojective algebra $\Pi_\lambda$ is a coadjoint orbit of the necklace Lie algebra $\mathbb{N}_Q$ whenever $\alpha$ is a minimal non-zero element of $\Sigma_\lambda$ the set of dimension vectors of simple representations of $\Pi_\lambda$.*

W. Crawley-Boevey has given a combinatorial description of the set $\Sigma_\lambda$ in [4]. We recover the Calogero case back and prove that the spaces appearing in a conjectural extension to arbitrary extended Dynkin quivers (conjectured by M. Holland and W. Crawley-Boevey) are all coadjoint orbits.

In the last two sections we try to explain why precisely these $(\alpha, \lambda)$ couples appear from the viewpoint of noncommutative geometry. As the path algebra $\mathbb{CQ}$ is a formally smooth algebra as in [7], its representation spaces $rep_\alpha\ \mathbb{Q}$ are smooth varieties and the noncommutative functions and differential forms induce invariant classical functions and forms on these varieties and their quotient varieties. On the other hand, we will show that the deformed preprojective algebra $\Pi_\lambda$ is *not* formally smooth and so should be viewed as a singular subvariety of the noncommutative manifold corresponding to $\mathbb{CQ}$. As such, the noncommutative vectorfields on $\mathbb{CQ}$ (the Lie algebra $\mathbb{N}_Q$) have rather unpredictable behavior on the singular closed subvariety $\Pi_\lambda$. However, for those dimension vectors $\alpha$ such that $rep_\alpha\ \Pi_\lambda$ is smooth (that is, $\Pi_\lambda$ is an $\alpha$-smooth subvariety of $\mathbb{CQ}$) things should work out. We conjecture that the corresponding $\alpha$ are precisely the minimal elements in $\Sigma_\lambda$ (the coadjoint orbits). We prove this for the preprojective algebra $\Pi_0$ and prove a variation (using hyper-Kähler reductions) for $\Pi_\lambda$. These results are based on the calculation of $Ext^1$'s of representations of $\Pi_0$ due to W. Crawley-Boevey [5].

## 2. Necklace Lie algebras.

In this section we introduce the main object of this note in a purely combinatorial way. Recall that a *quiver* $Q$ is a finite directed graph on a set of vertices $Q_v = \{v_1, \ldots, v_k\}$, having a finite set $Q_a = \{a_1, \ldots, a_l\}$ of arrows, where we allow loops as

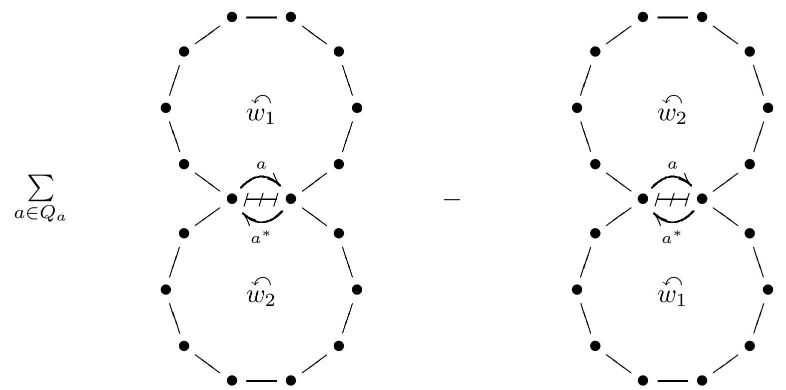



FIGURE 1. Lie bracket $[w_1, w_2]$ in $\mathbb{N}_Q$.

well as multiple arrows between vertices. An arrow $a$ with starting vertex $s(a) = v_i$ and terminating vertex $t(a) = v_j$ will be depicted as $\textcircled{j} \xleftarrow{a} \textcircled{i}$. The quiver information is encoded in the *Euler form* which is the bilinear form on $\mathbb{Z}^k$ determined by the matrix $\chi_Q \in M_k(\mathbb{Z})$ with

$$\chi_{ij} = \delta_{ij} - \# \{ \ a \in Q_a \ \mid \ \textcircled{j} \xleftarrow{a} \textcircled{i} \ \}$$

The symmetrization $T_Q = \chi_Q + \chi_Q^{tr}$ of this matrix determines the *Tits form* of the quiver $Q$. An oriented cycle $c = a_{i_u} \dots a_{i_1}$ of length $u \geq 1$ is a concatenation of arrows in $Q$ such that $t(a_{i_j}) = s(a_{i_{j+1}})$ and $t(a_{i_u}) = s(a_{i_1})$. In addition to these there are $k$ oriented cycles $e_i$ of length 0 corresponding to the vertices of $Q$. All oriented cycles $c'$ obtained from $c$ by cyclically permuting the arrow components are said to be equivalent to $c$. A *necklace word* $w$ for $Q$ is an equivalence class of oriented cycles in the quiver $Q$.

The *double quiver* $\mathbb{Q}$ of $Q$ is the quiver obtained by adjoining to every arrow (or loop) $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in $Q$ an arrow in the opposite direction $\textcircled{j} \xrightarrow{a^*} \textcircled{i}$. That is, $\chi_{\mathbb{Q}} = T_Q - \mathbb{1}_k$.

The *necklace Lie algebra* $\mathbb{N}_Q$ for the quiver $Q$ has as basis the set of all necklace words $w$ for the *double* quiver $\mathbb{Q}$ and where the Lie bracket $[w_1, w_2]$ is determined as in figure 1. That is, for every arrow $a \in Q_a$ we look for an occurrence of $a$ in $w_1$ and of $a^*$ in $w_2$. We then open up the necklaces by removing these factors and regluing the open ends together to form a new necklace word. We repeat this operation for *all* occurrences of $a$ (in $w_1$) and $a^*$ (in $w_2$). We then replace the roles of $a^*$ and $a$ and redo this operation with a minus sign. Finally, we add up all these obtained necklace words for all arrows $a \in Q_a$. Using this graphical description it is a pleasant exercise to verify the Jacobi identity for $\mathbb{N}_Q$.

## 3. AN ACYCLICITY RESULT.

The *path algebra* $\mathbb{C}Q$ of a quiver $Q$ has as basis the set of all oriented paths $p = a_{i_u} \dots a_{i_1}$ of length $u \geq 1$ in the quiver, that is $s(a_{i_{j+1}}) = t(a_{i_j})$ together with the vertex-idempotents $e_i$ of length zero. Multiplication in $\mathbb{C}Q$ is induced by (left) concatenation of paths. More precisely, $1 = e_1 + \dots + e_k$ is a decomposition of 1 into mutually orthogonal idempotents and further we define

- $e_j.a$ is always zero unless $\textcircled{j} \xleftarrow{a} \bigcirc$ in which case it is the path $a$,
- $a.e_i$ is always zero unless $\bigcirc \xleftarrow{a} \textcircled{i}$ in which case it is the path $a$,

- $a_i.a_j$ is always zero unless $\circ \xleftarrow{a_i} \circ \xleftarrow{a_j} \circ$ in which case it is the path $a_i a_j$.

Path algebras of quivers are the archetypical examples of *formally smooth algebras* as introduced and studied in [7].

In this section we will generalize Kontsevich's acyclicity result for the noncommutative deRham cohomology of the free algebra [14] to that of the path algebra $\mathbb{C}Q$. The crucial idea is to consider the *relative* differential forms (as defined in [7]) of $\mathbb{C}Q$ with respect to the semisimple subalgebra $V = \mathbb{C} \times \ldots \times \mathbb{C}$ generated by the vertex idempotents. The idea being that in considering quiver representations one works in the category of $V$-algebras rather than $\mathbb{C}$-algebras.

For a subalgebra $B$ of $A$, let $\overline{A}_B$ denote the cokernel of the inclusion as $B$-bimodule. The space of relative differential forms of degree $n$ of $A$ with respect to $B$ is

$$\Omega^n_B\ A = A \otimes_B \underbrace{\overline{A}_B \otimes_B \ldots \otimes_B \overline{A}_B}_{n}$$

The space $\Omega^\bullet_B\ A$ is given a differential graded algebra structure by taking the multiplication

$$(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_m) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_m)$$

and the differential $d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n)$, see [7]. Here, $(a_0, \ldots, a_n)$ is a representant of the class $a_0 da_1 \ldots da_n \in \Omega^n_B\ A$ and we recall that $\Omega^\bullet_B\ A$ s generated by the $a$ and $da$ for all $a \in A$. The *relative cohomology* $H^n_B\ A$ is defined as the cohomology of the complex $\Omega^\bullet_B\ A$.

For $\theta \in Der_B\ A$, the Lie algebra of $B$-derivations of $A$ (that is $\theta$ is a derivation of $A$ and $\theta(B) = 0$), we define a degree preserving derivation $L_\theta$ and a degree $-1$ super-derivation $i_\theta$ on $\Omega^\bullet_B\ A$

$$\begin{tikzcd}
\Omega^{n-1}_B\ A \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^n_B\ A \arrow[l, bend left, "i_\theta"] \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^{n+1}_B\ A \arrow[l, bend left, "i_\theta"] \arrow[loop below, "L_\theta"]
\end{tikzcd}$$

by the rules

$$\begin{cases} L_\theta(a) = \theta(a) & L_\theta(da) = d\ \theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

for all $a \in A$. We have the Cartan homotopy formula $L_\theta = i_\theta \circ d + d \circ i_\theta$ as both sides are degree preserving derivations on $\Omega^\bullet_B\ A$ and they agree on all the generators $a$ and $da$ for $a \in A$.

**Lemma 3.1.** *Let $\theta, \gamma \in Der_B\ A$, then we have on $\Omega^\bullet_B\ A$ the identities of operators*

$$\begin{cases} L_\theta \circ i_\gamma - i_\gamma \circ L_\theta = [L_\theta, i_\gamma] & = i_{[\theta,\gamma]} = i_{\theta\circ\gamma - \gamma\circ\theta} \\ L_\theta \circ L_\gamma - L_\gamma \circ L_\theta = [L_\theta, L_\gamma] & = L_{[\theta,\gamma]} = L_{\theta\circ\gamma - \gamma\circ\theta} \end{cases}$$

*Proof.* Consider the first identity. By definition both sides are degree $-1$ super-derivations on $\Omega^\bullet_B\ A$ so it suffices to check that they agree on generators. Clearly, both sides give 0 when evaluated on $a \in A$ and for $da$ we have

$$(L_\theta \circ i_\gamma - i_\gamma \circ L_\theta) da = L_\theta\ \gamma(a) - i_\gamma\ d\ \theta(a) = \theta\ \gamma(a) - \gamma\ \theta(a) = i_{[\theta,\gamma]}(da)$$

A similar argument proves the second identity. $\square$

Specialize to the quiver-case with $A = \mathbb{C}Q$ the path algebra and $B = V = \mathbb{C}^k$ the vertex algebra.

**Lemma 3.2.** *Let $Q$ be a quiver on $k$ vertices, then a basis for $\Omega^n_V \ \mathbb{C}Q$ is given by the elements*

$$p_0 dp_1 \dots dp_n$$

*where $p_i$ is an oriented path in the quiver such that length $p_0 \geq 0$ and length $p_i \geq 1$ for $1 \leq i \leq n$ and such that the starting point of $p_i$ is the endpoint of $p_{i+1}$ for all $1 \leq i \leq n-1$.*

*Proof.* Clearly $l(p_i) \geq 1$ when $i \geq 1$ or $p_i$ would be a vertex-idempotent whence in $V$. Let $v$ be the starting point of $p_i$ and $w$ the end point of $p_{i+1}$ and assume that $v \neq w$, then

$$p_i \otimes_V p_{i+1} = p_i v \otimes_V w p_{i+1} = p_i v w \otimes_V p_{i+1} = 0$$

from which the assertion follows. $\square$

**Proposition 3.3.** *Let $Q$ be a quiver on $k$ vertices, then the relative differential form-algebra $\Omega^\bullet_V \ \mathbb{C}Q$ is formal. In fact, the complex is acyclic*

$$\begin{cases} H^0_V \ \mathbb{C}Q & \simeq \mathbb{C} \times \dots \times \mathbb{C} \ (k \text{ factors}) \\ H^n_V \ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

*Proof.* Define the *Euler derivation* $E$ on $\mathbb{C}Q$ by the rules that

$$E(e_i) = 0 \ \forall \ 1 \leq i \leq k \qquad \text{and} \qquad E(a) = a \ \forall a \in Q_a$$

By induction on the length $l(p)$ of an oriented path $p$ in the quiver $Q$ one easily verifies that $E(p) = l(p)p$. By induction one can also proof that $L_E(p_0 dp_1 \dots dp_n) = l(p_0) + \dots + l(p_n)$. This implies that $L_E$ is a bijection on each $\Omega^i_V \ \mathbb{C}Q$, where $i > 1$ and on $\Omega^0_V \ \mathbb{C}Q$, $L_E$ has V as its kernel. By applying the Cartan homotopy formula for $L_E$, we obtain that the complex is acyclic. $\square$

The complex $\Omega^\bullet_V \ \mathbb{C}Q$ induces the *relative Karoubi complex*

$$\mathsf{dR}^0_V \ \mathbb{C}Q \xrightarrow{d} \mathsf{dR}^1_V \ \mathbb{C}Q \xrightarrow{d} \mathsf{dR}^2_V \ \mathbb{C}Q \xrightarrow{d} \dots$$

with

$$\mathsf{dR}^n_V \ \mathbb{C}Q = \frac{\Omega^n_V \ \mathbb{C}Q}{\sum_{i=0}^n [\ \Omega^i_V \ \mathbb{C}Q, \Omega^{n-i}_V \ \mathbb{C}Q\ ]}$$

In this expression the brackets denote supercommutators with respect to the grading on $\Omega^\bullet_V \ \mathbb{C}Q$. In the commutative case, $\mathsf{dR}^0$ are the functions on the manifold and $\mathsf{dR}^1$ the 1-forms.

**Lemma 3.4.** *A $\mathbb{C}$-basis for the noncommutative functions*

$$\mathsf{dR}^0_V \ \mathbb{C}Q \simeq \frac{\mathbb{C}Q}{[\ \mathbb{C}Q, \mathbb{C}Q\ ]}$$

*are the necklace words in the quiver $Q$.*

*Proof.* Let $\mathbb{W}$ be the $\mathbb{C}$-space spanned by all necklace words $w$ in $Q$ and define a linear map

$$\mathbb{C}Q \xrightarrow{n} \mathbb{W} \qquad \begin{cases} p \mapsto w_p & \text{if } p \text{ is a cycle} \\ p \mapsto 0 & \text{if } p \text{ is not} \end{cases}$$

for all oriented paths $p$ in the quiver $Q$, where $w_p$ is the necklace word in $Q$ determined by the oriented cycle $p$. Because $w_{p_1 p_2} = w_{p_2 p_1}$ it follows that the commutator subspace $[\mathbb{C}Q, \mathbb{C}Q]$ belongs to the kernel of this map. Conversely, let

$$x = x_0 + x_1 + \dots + x_m$$

be in the kernel where $x_0$ is a linear combination of non-cyclic paths and $x_i$ for $1 \leq i \leq m$ is a linear combination of cyclic paths mapping to the same necklace word $w_i$, then $n(x_i) = 0$ for all $i \geq 0$. Clearly, $x_0 \in [\mathbb{C}Q, \mathbb{C}Q]$ as we can write every noncyclic path $p = a.p' = a.p' - p'.a$ as a commutator. If $x_i = a_1p_1 + a_2p_2 + \ldots + a_lp_l$ with $n(p_i) = w_i$, then $p_1 = q.q'$ and $p_2 = q'.q$ for some paths $q, q'$ whence $p_1 - p_2$ is a commutator. But then, $x_i = a_1(p_1 - p_2) + (a_2 - a_1)p_2 + \ldots + a_lp_l$ is a sum of a commutator and a linear combination of strictly fewer elements. By induction, this shows that $x_i \in [\mathbb{C}Q, \mathbb{C}Q]$. ∎

**Lemma 3.5.** $\mathsf{dR}^1_V\ \mathbb{C}Q$ *is isomorphic as* $\mathbb{C}$*-space to*

$$\bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} v_i.\mathbb{C}Q.v_j\ da = \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} \textcircled{i} \cdots\cdots \textcircled{j}\ d\ \textcircled{j} \xleftarrow{a} \textcircled{i}$$

*Proof.* If $p.q$ is not a cycle, then $pdq = [p, dq]$ and so vanishes in $\mathsf{dR}^1_V\ \mathbb{C}Q$ so we only have to consider terms $pdq$ with $p.q$ an oriented cycle in $Q$. For any three paths $p, q$ and $r$ in $Q$ we have the equality

$$[p.qdr] = pqdr - qd(rp) + qrdp$$

whence in $\mathsf{dR}^1_V\ \mathbb{C}Q$ we have relations allowing to reduce the length of the differential part

$$qd(rp) = pqdr + qrdp$$

so $\mathsf{dR}^1_V\ \mathbb{C}Q$ is spanned by terms of the form $pda$ with $a \in Q_a$ and $p.a$ an oriented cycle in $Q$. Therefore, we have a surjection

$$\Omega^1_V\ \mathbb{C}Q \twoheadrightarrow \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} v_i.\mathbb{C}Q.v_j\ da$$

By construction, it is clear that $[\Omega^0_V\ \mathbb{C}Q, \Omega^1_{rel}\ \mathbb{C}Q]$ lies in the kernel of this map and using an argument as in the lemma above one shows also the converse inclusion. ∎

Using the above descriptions of $\mathsf{dR}^i_V\ \mathbb{C}Q$ for $i = 0, 1$ and the differential $\mathsf{dR}^0_V\ \mathbb{C}Q \xrightarrow{d} \mathsf{dR}^1_V\ \mathbb{C}Q$ we can define *partial differential operators* associated to any arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in $Q$.

$$\frac{\partial}{\partial a}\ :\ \mathsf{dR}^0_{rel}\ \mathbb{C}Q \longrightarrow v_i\mathbb{C}Qv_j \qquad \text{by} \qquad df = \sum_{a \in Q_a} \frac{\partial f}{\partial a} da$$

To take the partial derivative of a necklace word $w$ with respect to an arrow $a$, we run through $w$ and each time we encounter $a$ we open the necklace by removing that occurrence of $a$ and then take the sum of all the paths obtained.

Defining the *relative deRham cohomology* $H^n_{dR}\ \mathbb{C}Q$ to be the cohomology of the Karoubi complex and observing that the operators $L_\theta$ and $i_\theta$ on $\Omega^\bullet_V\ \mathbb{C}Q$ induce operators on the Karoubi complex, we have the *acyclicity result*

**Theorem 3.6.** *The relative Karoubi complex is acyclic. In particular,*

$$\begin{cases} H^0_{dR}\ \mathbb{C}Q & \simeq\ V \\ H^n_{dR}\ \mathbb{C}Q & \simeq\ 0 \qquad \forall n \geq 1 \end{cases}$$

## 4. Symplectic interpretation.

In this section we use the acyclicity result to give a Poisson interpretation to the Lie bracket in $\mathbb{N}_Q$. This generalizes the *Kontsevich bracket* [14] in the free case to path algebras of doubles of quivers. If $Q$ is a quiver with double quiver $\mathbb{Q}$, then we can define a canonical *symplectic structure* on the path algebra of the double $\mathbb{C}\mathbb{Q}$ determined by the element

$$\omega = \sum_{a \in Q_a} da^* da \in \mathsf{dR}^2_V\ \mathbb{C}\mathbb{Q}$$

As in the commutative case, $\omega$ defines a bijection between the noncommutative 1-forms $\mathsf{dR}^1_V\ \mathbb{C}\mathbb{Q}$ and the *noncommutative vectorfields* which are defined to be the $V$-derivations of $\mathbb{C}\mathbb{Q}$. This correspondence is

$$Der_V\ \mathbb{C}\mathbb{Q} \xrightarrow{\tau} \mathsf{dR}^1_V\ \mathbb{C}\mathbb{Q} \qquad \text{given by} \qquad \tau(\theta) = i_\theta(\omega)$$

In analogy with the commutative case we define a derivation $\theta \in Der_V\ \mathbb{C}\mathbb{Q}$ to be *symplectic* if and only if $L_\theta \omega = 0 \in \mathsf{dR}^2_V\ \mathbb{C}\mathbb{Q}$ and denote the subspace of symplectic derivations by $Der_\omega\ \mathbb{C}\mathbb{Q}$. It follows from the homotopy formula and the fact that $\omega$ is a closed form, that $\theta \in Der_\omega\ \mathbb{C}\mathbb{Q}$ implies $L_\theta\omega = di_\theta\omega = d\tau(\theta) = 0$. That is, $\tau(\theta)$ is a closed form which by the acyclicity of the Karoubi complex shows that it must be an exact form. That is we have an isomorphism of exact sequences of $\mathbb{C}$-vectorspaces

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "="'] & \mathsf{dR}^0_V\ \mathbb{C}\mathbb{Q} \arrow[r, "d"] \arrow[d, "\simeq"'] & (\mathsf{dR}^1_V\ \mathbb{C}\mathbb{Q})_{exact} \arrow[r] \arrow[d, "\tau^{-1}"'] & 0 \\
0 \arrow[r] & V \arrow[r] & \dfrac{\mathbb{C}\mathbb{Q}}{[\mathbb{C}\mathbb{Q}, \mathbb{C}\mathbb{Q}]} \arrow[r] & Der_\omega\ \mathbb{C}\mathbb{Q} \arrow[r] & 0
\end{tikzcd}$$

The symplectic structure $\omega$ defines a Poisson bracket on the noncommutative functions.

**Definition 4.1.** Let $Q$ be a quiver and $\mathbb{Q}$ its double. The *Kontsevich bracket* on the necklace words in $\mathbb{Q}$, $\mathsf{dR}^0_V\ \mathbb{C}\mathbb{Q}$ is defined to be

$$\{w_1, w_2\}_K = \sum_{a \in Q_a} \left(\frac{\partial w_1}{\partial a}\frac{\partial w_2}{\partial a^*} - \frac{\partial w_1}{\partial a^*}\frac{\partial w_2}{\partial a}\right) \text{ mod } [\mathbb{C}\mathbb{Q}, \mathbb{C}\mathbb{Q}]$$

By the description of the partial differential operators it is clear that $\mathsf{dR}^0_V\ \mathbb{C}\mathbb{Q}$ with this bracket is isomorphic to the necklace Lie algebra $\mathbb{N}_Q$.

The symplectic derivations $Der_\omega\ \mathbb{C}\mathbb{Q}$ have a natural Lie algebra structure by commutators of derivations. We will show that $\tau^{-1} \circ d$ is a Lie algebra morphism.

For every necklace word $w$ we have a symplectic derivation $\theta_w = \tau^{-1} dw$ defined by

$$\begin{cases} \theta_w(a) & = -\frac{\partial w}{\partial a^*} \\ \theta_w(a^*) & = \frac{\partial w}{\partial a} \end{cases}$$

With this notation we get the following interpretations of the Kontsevich bracket

$$\{w_1, w_2\}_K = i_{\theta_{w_1}}(i_{\theta_{w_2}}\omega) = L_{\theta_{w_1}}(w_2) = -L_{\theta_{w_2}}(w_1)$$

where the next to last equality follows because $i_{\theta_{w_2}}\omega = dw_2$ and the fact that $i_{\theta_{w_1}}(dw) = L_{\theta_{w_1}}(w)$ for any $w$. More generally, for any $V$-derivation $\theta$ and any necklace word $w$ we have the equation

$$i_\theta(i_{\theta_w}\omega) = L_\theta(w).$$

When we look at the image of the Kontsevich bracket under $\tau^{-1}d$, we obtain the following

$$\begin{aligned}\tau^{-1}d\{w_1,w_2\}_K &= \tau^{-1}dL_{\theta_{w_1}}w_2\\ &= \tau^{-1}L_{\theta_{w_1}}dw_2\\ &= \tau^{-1}L_{\theta_{w_1}}i_{\theta_{w_2}}\omega\\ &= \tau^{-1}([L_{\theta_{w_1}},i_{\theta_{w_2}}]+i_{\theta_{w_2}}L_{\theta_{w_1}})\omega\\ &= \tau^{-1}i_{[\theta_{w_1},\theta_{w_2}]}\omega\\ &= [\theta_{w_1},\theta_{w_2}]\end{aligned}$$

Above we made use of the fact that $L_\theta$ commutes with $d$, and the defining equation $dw_2 = i_{\theta_{w_2}}\omega$. In the fourth line we omitted the last term because $\theta_{w_1}$ is a symplectic derivation. Finally lemma 3.1 enabled us to transform the commutator in $i$ and $L$ to of commutator of the derivations $\theta_{w_1}$ and $\theta_{w_2}$. This calculation concluded the proof of :

**Theorem 4.2.** *With notations as before,* $\mathsf{dR}^0_{rel}\ \mathbb{C}Q^d$ *with the Kontsevich bracket is isomorphic to the necklace Lie algebra* $\mathbb{N}_Q$*, and the sequence*

$$0 \longrightarrow V \longrightarrow \mathbb{N}_Q \xrightarrow{\tau^{-1}d} Der_\omega\ \mathbb{CQ} \longrightarrow 0$$

*is an exact sequence (hence a central extension) of Lie algebras.*

## 5. Coadjoint orbits.

Consider a dimension vector $\alpha = (n_1,\ldots,n_k)$, that is, a $k$-tuple of natural numbers, then the space of $\alpha$-dimensional representations of the double quiver $\mathbb{Q}$, $rep_\alpha\ \mathbb{Q}$ can be identified via the trace pairing with the cotangent bundle $T^*\ rep_\alpha\ Q$ of the space of $\alpha$-dimensional representations of the quiver $Q$, see for example [4], and as such acquires a natural symplectic structure. The natural action of the basechange group $GL(\alpha) = GL_{n_1}\times\ldots\times GL_{n_k}$ on $rep_\alpha\ \mathbb{Q}$ is symplectic and induces a Poisson structure on the coordinate ring as well as on the ring of polynomial quiver invariants, which are generated by traces along oriented cycles by [17].

The symplectic derivations $Der_\omega\ \mathbb{CQ}$ correspond to the $V$-automorphisms of the path algebra of the double $\mathbb{CQ}$ preserving the *moment element*

$$m = \sum_{a\in Q_a}[a,a^*] \in \mathbb{CQ}$$

For this reason it is natural to consider the *complex moment map*

$$rep_\alpha\ \mathbb{Q} \xrightarrow{\mu_{\mathbb{C}}} M^0_\alpha(\mathbb{C}) \qquad V \mapsto \sum_{a\in Q_a}[V_a,V_{a^*}]$$

where $M^0_\alpha(\mathbb{C})$ is the subspace of $k$-tuples $(m_1,\ldots,m_k)\in M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$ such that $\sum_i tr(m_i) = 0$, that is $M^0_\alpha(\mathbb{C}) = Lie\ PGL(\alpha)$ where $PGL(\alpha) = GL(\alpha)/\mathbb{C}^*(\mathbb{1}_{n_1},\ldots,\mathbb{1}_{n_k})$.

For $\lambda = (\lambda_1,\ldots,\lambda_k)\in\mathbb{C}^k$ such that $\sum_i n_i\lambda_i = 0$ we consider the element $\underline{\lambda} = (\lambda_1\mathbb{1}_{n_1},\ldots,\lambda_k\mathbb{1}_{n_k})$ in $M^0_\alpha(\mathbb{C})$. The inverse image $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is a $GL(\alpha)$-closed affine subvariety of $rep_\alpha\ \mathbb{Q}$.

In [9] V. Ginzburg proved the following coadjointness result using the results of the preceding sections.

**Theorem 5.1** (Ginzburg)**.** *Assume that* $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ *is irreducible and that* $PGL(\alpha)$ *acts freely on* $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$*, then the quotient variety (the orbit space)*

$$\mu_{\mathbb{C}}^{-1}(\underline{\lambda})/GL(\alpha)$$

*is a coadjoint orbit for the necklace Lie algebra* $\mathbb{N}_Q$*.*

Using results of W. Crawley-Boevey [4] we will identify the situations $(\alpha, \lambda)$ satisfying the conditions of the theorem. For $\lambda \in \mathbb{C}^k$ as above, W. Crawley-Boevey and M. Holland introduced and studied the *deformed preprojective algebra*

$$\Pi_\lambda = \frac{\mathbb{C}\mathbb{Q}}{(m - \lambda)}$$

where $\lambda = \lambda_1 e_1 + \ldots + \lambda_k e_k \in \mathbb{C}\mathbb{Q}$. From [6] we recall that $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is the scheme of $\alpha$-dimensional representations $\underline{rep}_\alpha\ \Pi_\lambda$ of the deformed preprojective algebra $\Pi_\lambda$.

We recall the characterization due to V. Kac [10] of the dimension vectors of indecomposable representations of the quiver $Q$. To a vertex $v_i$ in which $Q$ has no loop, we define a *reflection* $\mathbb{Z}^k \xrightarrow{r_i} \mathbb{Z}^k$ by

$$r_i(\alpha) = \alpha - T_Q(\alpha, \epsilon_i)\epsilon_i$$

where $\epsilon_i = (\delta_{1i}, \ldots, \delta_{ki})$. The *Weyl group of the quiver* $Q$ $Weyl_Q$ is the subgroup of $GL_k(\mathbb{Z})$ generated by all reflections $r_i$.

A *root* of the quiver $Q$ is a dimension vector $\alpha \in \mathbb{N}^k$ such that $rep_\alpha\ Q$ contains indecomposable representations. All roots have connected support. A root is said to be

$$\begin{cases} \textit{real} & \text{if } \chi_Q(\alpha, \alpha) = 1 \\ \textit{imaginary} & \text{if } \chi_Q(\alpha, \alpha) \leq 0 \end{cases}$$

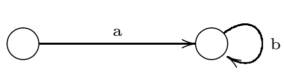
For a fixed quiver $Q$ we will denote the set of all roots, real roots and imaginary roots respectively by $\Delta, \Delta_{re}$ and $\Delta_{im}$. With $\Pi$ we denote the set $\{\epsilon_i \mid v_i \text{ has no loops }\}$. The *fundamental set of roots* is defined to be the following set of dimension vectors

$$F_Q = \{\alpha \in \mathbb{N}^k - \underline{0} \mid T_Q(\alpha, \epsilon_i) \leq 0 \text{ and } supp(\alpha) \text{ is connected }\}$$

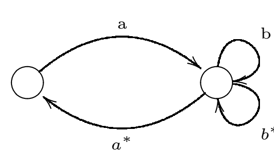
Kac's result asserts that

$$\begin{cases} \Delta_{re} & = Weyl_Q.\Pi \cap \mathbb{N}^k \\ \Delta_{im} & = Weyl_Q.F_Q \cap \mathbb{N}^k \end{cases}$$

**Example 5.2.** The quiver $Q$ and double quiver $\mathbb{Q}$ appearing in the study of Calogero phase space (see [20] and [8]) which stimulated the above generalizations are



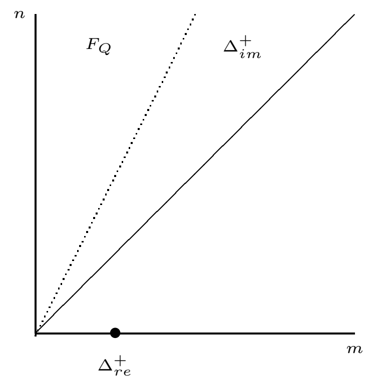

and




The Euler- and Tits form of the quiver $Q$ are determined by the matrices

$$\chi_Q = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad T_Q = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

The root-system for $Q$ is easy to work out. We have




$$\begin{cases} F_Q & = \{(m,n) \mid n \geq 2m\} \\ \Delta^+_{im} & = \{(m,n) \mid n \geq n\} \\ \Pi = \Delta^+_{re} & = \{(1,0)\} \end{cases}$$

Fix $\lambda \in \mathbb{C}^k$ and denote $\Delta^+_\lambda$ to be the set of positive roots $\beta = (b_1, \ldots, b_k)$ for $Q$ such that $\lambda.\beta = \sum_i \lambda_i b_i = 0$. With $S_\lambda$ (resp. $\Sigma_\lambda$) we denote the subsets of dimension vectors $\alpha$ which are roots for $Q$ such that

$$1 - \chi_Q(\alpha,\alpha) \quad \geq \text{ (resp. } >) \quad r - \chi_Q(\beta_1,\beta_1) - \ldots - \chi_Q(\beta_r,\beta_r)$$

for all decompositions $\alpha = \beta_1 + \ldots + \beta_r$ with the $\beta_i \in \Delta^+_\lambda$. The main results of [4] can be summarized into :

**Theorem 5.3** (W. Crawley-Boevey)**.**
1. *$\alpha \in S_0$ if and only if $\mu_{\mathbb{C}}$ is a flat morphism. In this case, $\mu_{\mathbb{C}}$ is also surjective.*
2. *$\alpha \in \Sigma_\lambda$ if and only if $\Pi_\lambda$ has a simple $\alpha$-dimensional representation. In this case, $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is a reduced and irreducible complete intersection of dimension $1 + \alpha.\alpha - 2\chi_Q(\alpha,\alpha)$.*

Using the results of [17] one verifies that the set of dimension vectors of simple representations of $\mathbb{Q}$ coincides with the fundamental set $F_Q$. As any simple $\Pi_\lambda$-representation is a simple $\mathbb{Q}$-representations it follows that $\Sigma_\lambda \hookrightarrow F_Q$.

**Example 5.4.** For the Calogero-example above, we have
1. The set $S_0$ consisting of all $(m,n)$ such that the complex moment map $\mu_{\mathbb{C}}$ is surjective and flat is the set of roots
$$S_0 = \{(m,n) \mid n \geq 2m-1\} \sqcup \{(1,0)\}$$
2. The set $\Sigma_0$ of dimension vectors $(m,n)$ of simple representations of the preprojective algebra $\Pi_0$ is the set of roots
$$\Sigma_0 = \{(m,n) \mid n \geq 2m\} \sqcup \{(1,0)\}$$
which is $F_Q \sqcup \{(1,0)\}$.
3. For $\lambda = (-n,m)$ with $gcd(m,n) = 1$, the set $\Sigma_\lambda$ of dimension vectors of simple representations of the deformed preprojective algebra is the set of roots
$$\Sigma_\lambda = \{k.(m,n) \mid k \in \mathbb{N}_+\}$$
with unique minimal element $(m,n)$.

For the first two parts the essential calculation is to verify the conditions on the decomposition $(m,n) = (m-1,n) + (1,0)$.

We obtain the following combinatorial description of the couples $(\alpha,\lambda)$ for which Ginzburg's criterium applies.

**Theorem 5.5.** *$\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is irreducible with a free action of $PGL(\alpha)$ (and hence $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})/GL(\alpha)$ is a coadjoint orbit for $\mathbb{N}_Q$) if and only if $\alpha$ is a minimal non-zero element of $\Sigma_\lambda$.*

*Proof.* We know that $\mu_{\mathbb{C}}^{-1}(\underline{\lambda}) = \underline{rep}_\alpha\ \Pi_\lambda$. By a result of M. Artin [1] one knows that the geometric points of the quotient scheme $\underline{rep}_\alpha\ \Pi_\lambda/GL(\alpha)$ are the isomorphism classes of $\alpha$-dimensional semi-simple representations of $\Pi_\lambda$. Moreover, the $PGL(\alpha)$-stabilizer of a point in $\underline{rep}_\alpha\ \Pi_\lambda$ is trivial if and only if it determines a simple $\alpha$-dimensional representation of $\Pi_\lambda$. The result follows from this and the results recalled above. □

**Example 5.6.** Consider the special case when $\lambda = (-n, 1)$ and $\alpha = (1, n)$ the unique minimal element in $\Sigma_\lambda$, then it follows from [20] that we have canonical identifications of the quotient varieties

$$iss_\alpha\ \Pi_\lambda \simeq Calo_n$$

where $Calo_n$ is the phase space of $n$ Calogero particles. In particular, $Calo_n$ is a coadjoint orbit. Wilson [20] has shown that

$$Gr^{ad} = \bigsqcup_n Calo_n$$

where $Gr^{ad}$ is the adelic Grassmannian which can be thought of as the space parametrizing isomorphism classes of right ideals in the first Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$ by [3]. In [2] it is shown that there is a non-differentiable action of the automorphism group of $A_1(\mathbb{C})$ on $Gr^{ad}$ having a transitive action on each of the $Calo_n$. It was then conjectured by Y. Berest and G. Wilson that $Calo_n$ might be a coadjoint orbit for a central extension of the automorphism group.

**Example 5.7.** M. Holland and W. Crawley-Boevey have a conjectural extension of the foregoing example. Let $Q'$ be an extended Dynkin quiver on $k$ vertices $\{v_1, \ldots, v_k\}$ with minimal imaginary root $\delta = (d_1, \ldots, d_k)$. A vertex $v_i$ is said to be an extending vertex provided $d_i = 1$. Consider the quiver $Q$ on $k+1$ vertices $\{v_0, v_1, \ldots, v_k\}$ which is $Q'$ on the last $k$ vertices and there is one extra arrow from $v_o$ to an extending vertex $v_i$. For a generic $\lambda' = (\lambda_1, \ldots, \lambda_k)$ they defined a noncommutative algebra $\mathcal{O}^{\lambda'}$ extending the role of the Weyl algebra in the previous example. They conjecture that there is a bijection between the isomorphism classes of stably free right ideals in $\mathcal{O}^\lambda$ and points in

$$\sqcup_n\ \mu_{\mathbb{C}}^{-1}(\lambda_n)/GL(\alpha_n)$$

where $\alpha_n = (1, n\delta)$ and $\lambda_n = (-n\lambda'.\delta, \lambda')$. This remains to be seen but from our theorem we deduce that each of the quotient varieties $\mu_{\mathbb{C}}^{-1}(\lambda_n)/GL(\alpha_n)$ is a coadjoint orbit for the necklace Lie algebra $\mathbb{N}_Q$.

If $\alpha \in \Sigma_\lambda$ but not minimal, there are several *representation types* $\tau = (m_1, \beta_1; \ldots, m_v, \beta_v)$ of semi-simple $\alpha$-dimensional representations of $\Pi_\lambda$ with the $\beta_i \in \Sigma_\lambda$ and $\sum m_i\beta_i = \alpha$ and the $m_i$ determine the multiplicities of the simple components. With $iss_\alpha(\tau)$ we denote the subvariety of the quotient variety $iss_\alpha\ \Pi_\lambda = rep_\alpha\ \Pi_\lambda/GL(\alpha)$ consisting of all semi-simple representations of type $\tau$.

Consider the algebra $A_Q = \mathbb{C}[\mathbb{N}_Q] \otimes_{\mathbb{C}} \mathbb{C}\mathbb{Q}$ which has a natural *trace map* $tr : A_Q \longrightarrow \mathbb{C}[\mathbb{N}_Q]$ mapping an oriented cycle in $\mathbb{Q}$ to the corresponding necklace word and all open paths to zero. With $Aut_Q$ we denote the automorphism group of trace preserving $\mathbb{C}$-algebra automorphisms of $A_Q$ which preserve the moment element $m = \sum_{a \in Q_a}[a, a^*]$. A natural extension of the above coadjoint orbit result would be a positive solution to the following problem.

**Question 5.8.** Does $Aut_Q$ act transitively on every stratum $iss_\alpha(\tau)$ ?

## 6. The smooth locus of $\Pi_0$.

In this section and the next we try to explain why exactly the couples $(\alpha, \lambda)$ with $\lambda.\alpha = 0$ and $\alpha$ a minimal non-zero element of $\Sigma_\lambda$ give rise to coadjoint orbits.

The path algebra $\mathbb{C}\mathbb{Q}$ of the double quiver $\mathbb{Q}$ is formally smooth in the sense of [7], that is, it has the lifting property with respect to nilpotent ideals. Hence, $\mathbb{C}\mathbb{Q}$ is the coordinate ring of a noncommutative affine manifold and has a good theory of differential forms (acyclicity).

On the other hand, we will see that the deformed preprojective algebras $\Pi_\lambda$ are *never* formally smooth. For this reason, the differential forms of $\mathbb{C}\mathbb{Q}$ when restricted to $\Pi_\lambda$ may have rather unpredictable behavior.

Still, it may be possible that certain representation spaces $rep_\alpha\ \Pi_\lambda$ are smooth and we need a notion of noncommutative (formal) smoothness depending on the dimension vector $\alpha$. This notion is Cayley-smoothness as introduced by C. Procesi in [19] and studied in detail in [16].

Let $\alpha = (n_1, \ldots, n_k)$ and set $n = \sum_i n_i$. With $\mathsf{alg}\ @_\alpha$ we denote the category of all $V$-algebras $A$ equipped with a trace map $tr : A \longrightarrow A$ (that is, such that for all $a, b \in A$ we have $tr(a)b = btr(a)$, $tr(ab) = tr(ba)$ and $tr(tr(a)b) = tr(a)tr(b)$) satisfying $tr(1) = n$ and the formal Cayley-Hamilton identity of degree $n$, see [19] such that $tr(e_i) = n_i$. Morphisms in $\mathsf{alg}\ @_\alpha$ are trace preserving algebra morphisms. An $\alpha$-*Cayley smooth* algebra $A$ is an algebra in $\mathsf{alg}\ @_\alpha$ having the lifting property with respect to nilpotent ideals in $\mathsf{alg}\ @_\alpha$. That is, every diagram

$$\begin{tikzcd} B \arrow[r, two heads, "\pi"] & \frac{B}{I} \\ & A \arrow[u, "\phi"'] \arrow[ul, dotted, "\exists\tilde{\phi}"] \end{tikzcd}$$

with $B, \frac{B}{I}$ in $\mathsf{alg}\ @_\alpha$, $I$ a nilpotent ideal and $\pi$ and $\phi$ trace preserving maps, can be completed with a trace preserving algebra map $\tilde{\phi}$. It is proved in [16] that $A$ is $\alpha$-Cayley smooth if and only if the scheme $\underline{rep}_\alpha\ A$ of $\alpha$-dimensional representations of $A$ is a smooth $GL(\alpha)$-variety.

In particular, if $(\lambda, \alpha)$ is such that $\lambda.\alpha = 0$ and $\alpha$ is a minimal non-zero vector in $\Sigma_\lambda$, then the level $\alpha$ approximation $\Pi_\lambda\ @_\alpha$ (which is the ring of $GL(\alpha)$-equivariant maps from $\underline{rep}_\alpha\ \Pi_\lambda$ to $M_n(\mathbb{C})$ with the induced trace from $M_n(\mathbb{C}[\underline{rep}_\alpha\ \Pi_\lambda))$ is $\alpha$-Cayley smooth. In fact, $\Pi_\lambda\ @_\alpha$ is an Azumaya algebra over the coadjoint orbit. A neat explanation for the description of the coadjoint orbits would be provided by a positive solution to the following problem.

**Question 6.1.** Conversely, if $\Pi_\lambda\ @_\alpha$ is $\alpha$-Cayley smooth, does it follow that $\alpha$ is a minimal non-zero vector in $\Sigma_\lambda$ ? More generally, does the $\alpha$-smooth locus of $\Pi_\lambda\ @_\alpha$, that is the locus $Sm_\alpha\ \Pi_\lambda$ in $\underline{iss}_\alpha\ \Pi_\lambda$ such that $\underline{rep}_\alpha\ \Pi_\alpha$ is smooth along $\pi^{-1}(Sm\ \Pi_\lambda)$, coincide with the Azumaya locus ?

We will give an affirmative solution in the special case of the preprojective algebra $\Pi_0$. By a result of W. Crawley-Boevey [5], we can control the $Ext^1$-spaces of representations of $\Pi_0$. Let $V$ and $W$ be representations of $\Pi_0$ of dimension vectors $\alpha$ and $\beta$, then we have

$$dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(V, W) = dim_{\mathbb{C}}\ Hom_{\Pi_0}(V, W) + dim_{\mathbb{C}}\ Hom_{\Pi_0}(W, V) - T_Q(\alpha, \beta)$$

For $\xi \in \underline{iss}_\alpha\ \Pi_0$ to belong to the smooth locus $\xi \in Sm_\alpha\ \Pi_0$ it is necessary and sufficient that $\underline{rep}_\alpha\ \Pi_0$ is smooth along the orbit $\mathcal{O}(M_\xi)$ where $M_\xi$ is the semi-simple $\alpha$-dimensional representation of $\Pi_0$ corresponding to $\xi$.

Assume that $\xi$ is of type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$, that is,

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z}$$

with $S_i$ a simple $\Pi_0$-representation of dimension vector $\alpha_i$. Then, the normal space to the orbit $\mathcal{O}(M_\xi)$ is determined by $Ext^1_{\Pi_o}(M_\xi, M_\xi)$ and can be depicted by a local quiver setting $(Q_\xi, \alpha_\xi)$ where $Q_\xi$ is a quiver on $z$ vertices having as many arrows from vertex $i$ to vertex $j$ as the dimension of $Ext^1_{\Pi_0}(S_i, S_j)$ and where $\alpha_\xi = \alpha_\tau = (e_1, \ldots, e_z)$. Applying the Luna slice theorem [18] we have

**Lemma 6.2.** *With notations as above, $\xi \in Sm_\alpha\ \Pi_0$ if and only if*

$$dim\ GL(\alpha) \times^{GL(\alpha_\xi)} Ext^1_{\Pi_0}(M_\xi, M_\xi) = dim_{M_\xi}\ \underline{rep}_\alpha\ \Pi_0$$

As we have enough information to compute both sides, we can prove :

**Theorem 6.3.** *If $\xi \in \underline{iss}_\alpha\ \Pi_0$ with $\alpha = (a_1, \ldots, a_k) \in S_{\underline{0}}$, then $\xi \in Sm_\alpha\ \Pi_0$ if and only if $M_\xi$ is a simple $n$-dimensional representation of $\Pi_0$. That is, the smooth locus of $\Pi_0$ coincides with the Azumaya locus.*

*Proof.* Assume that $\xi$ is a point of semi-simple representation type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$, that is,

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z} \qquad \text{with} \qquad dim(S_i) = \alpha_i$$

and $S_i$ a simple $\Pi_0$-representation. We have

$$\begin{cases} dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(S_i, S_j) & = -T_Q(\alpha_i, \alpha_j) \qquad i \neq j \\ dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(S_i, S_i) & = 2 - T_Q(\alpha_i, \alpha_i) \end{cases}$$

But then, the dimension of $Ext^1_{\Pi_0}(M_\xi, M_\xi)$ is equal to

$$\sum_{i=1}^{z} (2 - T_Q(\alpha_i, \alpha_i)) e_i^2 + \sum_{i \neq j} e_i e_j (-T_Q(\alpha_i, \alpha_j) = 2 \sum_{i=1}^{z} e_i - T_Q(\alpha, \alpha)$$

from which it follows immediately that

$$dim\ GL(\alpha) \times^{GL(\alpha_\xi)} Ext^1_{\Pi_0}(M_\xi, M_\xi) = \alpha.\alpha + \sum_{i=1}^{z} e_i^2 - T_Q(\alpha, \alpha)$$

On the other hand, as $\alpha \in S_{\underline{0}}$ we know that

$$dim\ \underline{rep}_\alpha\ \Pi_0 = \alpha.\alpha - 1 + 2p_Q(\alpha) = \alpha.\alpha - 1 + 2 - 2\chi_Q(\alpha, \alpha) = \alpha.\alpha + 1 - T_Q(\alpha, \alpha)$$

But then, equality occurs if and only if $\sum_i e_i^2 = 1$, that is, $\tau = (1, \alpha)$ or $M_\xi$ is a simple $n$-dimensional representation of $\Pi_0$. $\square$

In particular it follows that the preprojective algebra $\Pi_0$ is *never* formally smooth as this implies that all the representation varieties must be smooth. Further, as $\vec{v_i} = (0, \ldots, 1, 0, \ldots, 0)$ are dimension vectors of simple representations of $\Pi_0$ it follows that $\Pi_0$ is $\alpha$-smooth if and only if $\alpha = \vec{v_i}$ for some $i$.

**Example 6.4.** Let $Q$ be an extended Dynkin diagram and $\delta$ the minimal imaginary root, then $\delta \in S_0$. The dimension of the quotient variety

$$\begin{aligned} dim\ iss_\delta\ \Pi_0 &= dim\ rep_\delta\ \Pi_0 - \delta.\delta + 1 \\ &= 2 \end{aligned}$$

so it is a surface. The only other semi-simple $\delta$-dimensional representation of $\Pi_0$ is the trivial representation. By the theorem, this must be an isolated singular point of $iss_\delta\ Q$. In fact, one can show that $iss_\delta\ \Pi_0$ is the Kleinian singularity corresponding to the extended Dynkin diagram $Q$.

## 7. A sheaf of algebras.

We will prove that $\alpha$-Cayley smoothness of a closely related sheaf of algebras is equivalent to $\alpha$ being a minimal non-zero vector of $\Sigma_\lambda$. Recall that $rep_\alpha\ \mathbb{Q}$ admits a hyper-Kähler structure (that is, an action of the quaternion algebra $\mathbb{H} = \mathbb{R}.1 \oplus \mathbb{R}.i \oplus \mathbb{R}.j \oplus \mathbb{R}.k$) defined for all arrows $a \in Q_a$ and all arrows $b \in \mathbb{Q}_a$ by the formulae, see for example [5]

$$\begin{array}{ll} (i.V)_b = iV_b & \\ (j.V)_a = -V_{a^*}^\dagger & (j.V)_{a^*} = V_a^\dagger \\ (k.V)_a = -iV_{a^*}^\dagger & (k.V)_{a^*} = iV_a^\dagger \end{array}$$

where this time we denote the Hermitian adjoint of a matrix $M$ by $M^\dagger$ to distinguish it from the star-operation on the arrows of the double quiver $\mathbb{Q}$. Let $U(\alpha)$ be the product of unitary groups $U_{n_1} \times \ldots \times U_{n_k}$ and consider the *real moment map*

$$rep_\alpha\ \mathbb{Q} \xrightarrow{\mu_{\mathbb{R}}} Lie\ U(\alpha) \quad V \mapsto \sum_{\substack{\circ \xleftarrow{b} \circ \\ b \in \mathbb{Q}_a}} \frac{i}{2}[V_b, V_b^\dagger]$$

For $\lambda \in \mathbb{R}^k$, multiplication by the quaternion-element $h = \frac{i+k}{\sqrt{2}}$ gives a homeomorphism between the real varieties

$$\mu_{\mathbb{C}}^{-1}(\underline{\lambda}) \cap \mu_{\mathbb{R}}^{-1}(\underline{0}) \xrightarrow{h.} \mu_{\mathbb{C}}^{-1}(\underline{0}) \cap \mu_{\mathbb{R}}^{-1}(i\underline{\lambda})$$

Moreover, the hyper-Kähler structure commutes with the base-change action of $U(\alpha)$, whence we have a natural one-to-one correspondence between the quotient spaces

$$(\mu_{\mathbb{C}}^{-1}(\underline{\lambda}) \cap \mu_{\mathbb{R}}^{-1}(\underline{0}))/U(\alpha) \xrightarrow{h.} (\mu_{\mathbb{C}}^{-1}(\underline{0}) \cap \mu_{\mathbb{R}}^{-1}(i\underline{\lambda}))/U(\alpha)$$

see [5] for more details. By results of Kempf and Ness [12] we can identify the left hand side as the quotient variety $iss_\alpha\ \Pi_\lambda$ and by results of A. King [13] we can identify the right hand side as the moduli space $M_\alpha^{ss}(\Pi_0, \lambda)$ of $\lambda$-semistable $\alpha$-dimensional representations of the preprojective algebra $\Pi_0$. Recall that a representation $V \in rep_\alpha\ \mathbb{Q}$ is said to be $\lambda$-(semi)stable if and only if for every proper subrepresentation $W$ of $V$ say with dimension vector $\beta$ we have $\lambda.\beta > 0$ (resp. $\lambda.\beta \geq 0$). The scheme $\underline{rep}_\alpha^{ss}(\Pi_0, \lambda)$ of $\lambda$-semistable $\alpha$-dimensional representations of $\Pi_0$ is the intersection of $\mu_{\mathbb{C}}^{-1}(\underline{0})$ with the subvariety of $\lambda$-semistable representations in $rep_\alpha\ \mathbb{Q}$. The corresponding moduli space $M_\alpha^{ss}(\Pi_0, \lambda)$ classifies isomorphism classes of direct sums of $\lambda$-stable representations of $\Pi_0$ of total dimension $\alpha$. In view of the explicit form of the hyper-Kähler structure it follows that the deformed preprojective algebra $\Pi_\lambda$ has semi-simple representations of dimension vector $\alpha$ of representation type $\tau = (e_1, \beta_1; \ldots; e_r, \beta_r)$ if and only if the preprojective algebra $\Pi_0$ has $\lambda$-stable representations of dimension vectors $\beta_i$ for all $1 \leq i \leq r$. In particular, $\Pi_\theta$ has a simple representation of dimension vector $\alpha$ if and only if $\Pi_0$ has a $\theta$-stable representation of dimension vector $\alpha$.

Taking locally the algebras of $GL(\alpha)$-equivariant maps from $\underline{rep}_\alpha^{ss}(\Pi_0, \lambda)$ to $M_n(\mathbb{C})$ defines a sheaf of algebras in $\mathsf{alg}$ @$_\alpha$, $\mathcal{A}_{\lambda,\alpha}$ on the moduli space $M_\alpha^{ss}(\Pi_0, \lambda)$. The main result of this section asserts the following.

**Theorem 7.1.** *With notations as above, for $\alpha \in \Sigma_\lambda$ the following are equivalent :*

1. *$\mathcal{A}_{\lambda,\alpha}$ is a sheaf of $\alpha$-Cayley smooth algebras on the moduli space $M_\alpha^{ss}(\Pi_0, \lambda)$.*
2. *$\alpha$ is a minimal non-zero vector in $\Sigma_\lambda$ (and hence the quotient variety $iss_\alpha\ \Pi_\lambda$ is a coadjoint orbit for the necklace Lie algebra $\mathbb{N}_Q$).*

*Proof.* As $\alpha \in \Sigma_\lambda$ we know that $iss_\alpha\ \Pi_\lambda$ has dimension $1 + \alpha.\alpha - 2\chi_Q(\alpha, \alpha) - dim\ PGL(\alpha)$ which is equal to $2 - T_Q(\alpha, \alpha)$. By the hyper-Kähler correspondence

so is the dimension of $M^{ss}_\alpha(\Pi_0,\lambda)$, whence the open subset of $\mu_{\mathbb{C}}^{-1}(\underline{0})$ consisting of $\lambda$-semistable representations has dimension

$$1+\alpha.\alpha-2\chi_Q(\alpha,\alpha)$$

as there are $\lambda$-stable representations in it (again via the hyper-Kähler correspondence). Take a $GL(\alpha)$-closed orbit $\mathcal{O}(V)$ in this open set. That is, $V$ is the direct sum of $\lambda$-stable subrepresentations

$$V=S_1^{\oplus e_1}\oplus\ldots\oplus S_r^{\oplus e_r}$$

with $S_i$ a $\lambda$-stable representation of $\Pi_0$ of dimension vector $\beta_i$ occurring in $V$ with multiplicity $e_i$ whence $\alpha=\sum_i e_i\beta_i$.

Again, the normal space in $V$ to $\mathcal{O}(V)$ can be identified with $Ext^1_{\Pi_0}(V,V)$. As all $S_i$ are $\Pi_0$-representations we can determine this space by the knowledge of all $Ext^1_{\Pi_0}(S_i,S_j)$.

$$Ext^1_{\Pi_0}(S_i,S_j)=2\delta_{ij}-T_Q(\beta_i,\beta_j)$$

But then the dimension of the normal space to the orbit is

$$dim\ Ext^1_{\Pi_0}(V,V)=2\sum_{i=1}^r e_i-T_Q(\alpha,\alpha)$$

By the Luna slice theorem [18], the étale local structure in the smooth point $V$ is of the form $GL(\alpha)\times^{GL(\tau)}Ext^1(V,V)$ where $\tau=(e_1,\ldots,e_r)$ and is therefore of dimension

$$\alpha.\alpha+\sum_{i=1}^2 e_i^2-T_Q(\alpha,\alpha)$$

This number must be equal to the dimension of the subvariety of $\lambda$-semistable representations of $\Pi_0$ which has dimension $1+\alpha.\alpha-T_Q(\alpha,\alpha)$ if and only if $r=1$ and $e_1=1$, that is if and only if $V$ is $\lambda$-stable. Hence, if $rep^{ss}_\alpha(\Pi_0,\lambda)$ is smooth, then $\alpha$ must be a minimal non-zero vector in the set of dimension vectors of $\lambda$-stable representations of $\Pi_0$ and hence by the hyper-Kähler correspondence, $\alpha$ is a minimal non-zero vector in $\Sigma_\lambda$.

Conversely, if $\alpha$ is a minimal vector in $\Sigma_\lambda$, then $iss_\alpha\ \Pi_\lambda$ is a coadjoint orbit, whence smooth and hence so is $M^{ss}_\alpha(\Pi_0,\lambda)$ by the correspondence. Moreover, all $\alpha$-dimensional $\lambda$-semistable representations must be $\lambda$-stable by the minimality assumption and so $rep^{ss}_\alpha(\Pi_0,\lambda)$ is a principal $PGL(\alpha)$-fibration over $M^{ss}_\alpha(\Pi_0,\lambda)$ whence smooth. Therefore, $\mathcal{A}_{\lambda,\alpha}$ is a sheaf of $\alpha$-Cayley smooth algebras. □

Question 6.1 can be proved as in the case of $\Pi_0$ provided we know that

$$\begin{aligned}dim_{\mathbb{C}}\ Ext^1_{\Pi_\lambda}(M,N)=dim_{\mathbb{C}}\ Hom_{\Pi_\lambda}(M,N)+dim_{\mathbb{C}}\ Hom_{\Pi_\lambda}(N,M)\\ -T_Q(dim\ M,dim\ N)\end{aligned}$$

for all (simple) $\Pi_\lambda$ representations $M$ and $N$. Still, we can prove that $\Pi_\lambda$ can never be formally smooth and even that certain representation varieties are not smooth.

**Proposition 7.2.** *Let $\alpha\in\Sigma_\lambda$ such that $2\alpha\in\Sigma_\lambda$. Then, $\underline{rep}_{2\alpha}\ \Pi_\lambda$ is not smooth. In particular, $\Pi_\lambda$ is not formally smooth.*

*Proof.* As $\alpha\in\Sigma_\lambda$ we know that the local quiver $Q_\xi$ in a simple representation $S$ corresponding to $\xi$ is a one vertex quiver having $2-T_Q(\alpha,\alpha)$ loops. That is,

$$dim\ Ext^1_{\Pi_\lambda}(S,S)=2-T_Q(\alpha,\alpha)$$

But then, for $\xi\in\underline{iss}_{2\alpha}\ \Pi_\lambda$ a point corresponding to $S\oplus S$, the local quiver is still $Q_\xi$ but this time the local dimension vector $\alpha_\xi=2$. If $\xi$ lies in the smooth locus, then by the Luna slice theorem we must have

$$dim\ GL(2\alpha)\times^{GL_2}rep_{\alpha_\xi}\ Q_\xi=dim\ rep_{2\alpha}\ \Pi_\lambda$$

The left hand side is $4\alpha.\alpha + 4 - 4T_Q(\alpha,\alpha)$ whereas the right hand side is equal to (because $2\alpha \in \Sigma_\lambda$) $4\alpha.\alpha + 1 - 4T_Q(\alpha,\alpha)$, a contradiction. □

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